International Journal of Engineering, Science and Mathematics
Vol. 6 Issue 8 Decembert2017, ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Interval Oscillation Criteria for Impulsive Partial Differential Equations

V. Sadhasivam [1], **T. Raja** [2] and **K. Logaarasi** [3]
[1,2,3]PG and Research Department of Mathematics,
Thiruvalluvar Government Arts College,
(Affiliated to Periyar University, Salem - 636 011)
Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, India.

Abstract

In this paper, we obtain some sufficient conditions for the oscillations of all solutions of impulsive partial differential equations. The results gained here are based on the improvement impulses, delay and forcing term in the sequence of subintervals of $\mathbb{R}_+$ which develops some well-known results for the equations without impulses and the equations without delay. The main result is illustrate with a suitable example.

Keywords:

Oscillation; Impulse; Partial differential Equations; Forcing term.

2010 Mathematics Subject Classification:
35B05, 35L70, 35R10, 35R12.

Author correspondence:

Dr.V. Sadhasivam
Associate Professor and Head,
PG and Research Department of Mathematics,
Thiruvalluvar Government Arts College,
(Affiliated to Periyar University, Salem - 636 011)
Rasipuram - 637 401, Namakkal (Dt), Tamil Nadu, India.

1. Introduction

In recent years the theory of impulsive differential equations emerge as an important area of research, since such equations have applications in the control theory, physics, biology, population dynamics, economics, etc.

In [4], the problem of oscillation and non-oscillation of impulsive delay equation of the form

$$x''(t) + p(t)x(t-\rho) = e(t), \quad t \neq t_k,$$
$$x(t_k^+) = \alpha_k x(t_k), \quad x'(t_k^+) = \beta_k x'(t_k), \quad k = 1,2,\cdots$$

was studied by Huang using Kartsatos technique in the year 2006. Using the same approach in [20], Zhang et.al., considered the oscillation of second order forced FDE with impulses

$$x''(t) + p(t)f\big(x(t-\rho)\big) = e(t), \quad t \neq t_k,$$
$$x(t_k^+) = \alpha_k x(t_k), \quad x'(t_k^+) = \beta_k x'(t_k), \quad k = 1,2,\cdots$$

and established some interval oscillation criteria which developed some known results for the equations without delay or impulses [2, 9].

In the last decades, interval oscillation of impulsive differential equations was arousing the interest of many researchers, see [3, 8, 10-12, 14, 17, 18] and the references cited therein. For further details applications, one can refer the monographs [1, 7, 16, 19] and reference cited therein. Most of the existing literature concentrated on interval oscillation criteria for case of without delay and only very few papers appeared for case of with delay.

As far as author knowledge, it seems that there has been no paper dealing with interval oscillation criteria for impulsive partial differential equations. Motivated by this gap, we consider the following impulsive partial differential equations of the form

$$\left.\begin{aligned} &\frac{\partial}{\partial t}\left[r(t)g\left(\frac{\partial}{\partial t}u(x,t)\right)\right]+q(x,t)f(u(x,t-\rho))+\sum_{j=1}^{m}q_j(x,t)f_j(u(x,t-\rho))\\ &=a(t)\Delta u(x,t)+\sum_{s=1}^{l}a_s(t)\Delta u(x,t-\tau_s)+F(x,t),\quad t\neq t_k,\\ &u(x,t_k^+)=a_k(x,t_k,u(x,t_k))\\ &u_t(x,t_k^+)=b_k(x,t_k,u_t(x,t_k)),\quad k=1,2,\cdots,\quad (x,t)\in\Omega\times\mathbb{R}_+\equiv G,\end{aligned}\right\}\tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary ∂Ω, Δ is the Laplacian in the Euclidean space $\mathbb{R}^N$ and $\mathbb{R}_+=[0,+\infty)$.

Equation (1.1) is enhancement with one of the subsequent Dirichlet boundary condition,

$$u(x,t)=0,\qquad (x,t)\in\partial\Omega\times\mathbb{R}_+.\tag{1.2}$$

In the sequel, we assume that the following hypotheses (H) hold:

(H_1) $r(t)\in C^1(\mathbb{R}_+,(0,+\infty))$, $q(x,t),q_j(x,t)\in C(\bar{G},\mathbb{R}_+)$, $q(t)=\min_{x\in\bar{\Omega}}q(x,t)$, $q_j(t)=\min_{x\in\bar{\Omega}}q_j(x,t)$, $j=1,2,\cdots,m$, $f,f_j\in C(\mathbb{R},\mathbb{R})$ are convex in $\mathbb{R}_+$ with $uf(u)>0$, $uf_j(u)>0$ and $\frac{f(u)}{u}\geq\varepsilon>0$, $\frac{f_j(u)}{u}\geq\varepsilon_j>0$ for $u\neq0$, $j=1,2,\cdots,m$, $t-\rho<t$, $t-\tau_s<t$, $\lim_{t\to+\infty}t-\rho=\lim_{t\to+\infty}t-\tau_s=+\infty$, $s=1,2,\cdots,l$ and $F\in C(\bar{G},\mathbb{R})$.

(H_2) $g\in C(\mathbb{R},\mathbb{R})$ are convex in $\mathbb{R}_+$ with $ug(u)>0$, $g(u)\leq\eta u$ for $u\neq0$, $g^{-1}\in C(\mathbb{R},\mathbb{R})$ are continuous functions with $ug^{-1}(u)>0$ for $u\neq0$ and there exist positive constant ζ such that $g^{-1}(uv)\leq\zeta g^{-1}(u)g^{-1}(v)$ for $uv\neq0$ and $\int_{t_0}^{+\infty}g^{-1}\left(\frac{1}{r(s)}\right)ds=+\infty$.

(H_3) $a(t)$, $a_s(t)\in PC(\mathbb{R}_+,\mathbb{R}_+)$, $s=1,2,\dots,l$, where PC represents the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k$, $k=1,2,\cdots$, and left continuous at $t=t_k$, $k=1,2,\cdots$.

(H_4) $u(x,t)$ and its derivative $u_t(x,t)$ are piecewise continuous in t with discontinuities of firstkind only at $t=t_k$, $k=1,2,\cdots$, and left continuous at $t=t_k$, $u(x,t_k)=u(x,t_k^-)$, $u_t(x,t_k)=u_t(x,t_k^-)$, $k=1,2,\cdots$.

(H_5) $a_k,b_k\in PC(\bar{\Omega}\times\mathbb{R}_+\times\mathbb{R},\mathbb{R})$, $k=1,2,\cdots$,and there exist positive constants $\alpha_k,\alpha_k^*,\beta_k,\beta_k^*$such that $\alpha_k^*\leq\alpha_k\leq\beta_k^*\leq\beta_k$for $k=1,2,\cdots$,

$$\alpha_k^*\leq\frac{a_k(x,t_k,u(x,t_k))}{u(x,t_k)}\leq\alpha_k,\quad \beta_k^*\leq\frac{b_k(x,t_k,u_t(x,t_k))}{u_t(x,t_k)}\leq\beta_k.$$

Definition 1.1 [19]. A solution u of the problem (1.1)-(1.2) is a function $u\in C^2(\bar{\Omega}\times t_{-1},+\infty),\mathbb{R})\cap C(\bar{\Omega}\times\hat{t}_{-1},+\infty),\mathbb{R})$ that satisfies (1.1), where

$$t_{-1}:=\min\left\{0,\min_{1\leq s\leq l}\left\{\inf_{t\geq0}t-\tau_s\right\}\right\},\quad \hat{t}_{-1}:=\min\left\{0,\inf_{t\geq0}t-\rho\right\}.$$

Definition 1.2. The solution u of the problem (1.1)-(1.2) is said to be oscillatory in the domain G if it has arbitrary large zeros. Otherwise it is non-oscillatory.

It is identified that [15] the smallest eigenvalue $\lambda_0>0$ of the eigenvalue problem

$$\Delta\omega(x) + \lambda\omega(x) = 0, \quad \text{in}\,\Omega$$
$$\omega(x) = 0, \quad \text{on}\,\partial\Omega,$$

and the consequent eigenfunction $\Phi(x) > 0$ in Ω.

For convenience, we introduce the following notations:

$$v(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx, \quad Q(t) = \varepsilon q(t) + \sum_{j=1}^{m} \varepsilon_j q_j(t) \text{where} K_\Phi = \left(\int_\Omega \Phi(x)dx\right)^{-1}.$$

This paper is organized as follows: The main results are given in Section 2. In Section 3, one example is considered to illustrate the main result.

2. Main Result

In this section, the intervals $[c_1, d_1]$ and $[c_2, d_2]$ are considered to establish oscillation criteria. So we also assume that

(H_6) $c_i, d_i \notin \{t_k\}$, $i = 1,2$, $k = 1,2,\cdots$, with $c_1 < d_1$, $c_2 < d_2$ and $r(t) \geq 0$, $q(t) \geq 0$, $q_j(t) \geq 0, j = 1,2,\cdots,m$ for $t \in [c_1 - \rho, d_1] \cup [c_2 - \rho, d_2]$ and $F(t)$ has different signs in $[c_1 - \rho, d_1]$ and $[c_2 - \rho, d_2]$, for instance, let

$$F(t) \leq 0 \quad \text{for} t \in [c_1 - \rho, d_1], \quad \text{and} F(t) \geq 0 \quad \text{for} t \in [c_2 - \rho, d_2].$$

Denote

$$I(s) := \max\{i:\ t_0 < t_i < s\}, \quad r_i := \max\{r(t):\ t \in [c_i, d_i]\}, \quad i = 1,2$$
$$J_p(c_i, d_i) = \{p \in C^1[c_i, d_i], \quad p(t) \neq 0, \quad p(c_i) = p(d_i) = 0, \quad i = 1,2\}$$
$$J_G(c_i, d_i) = \{G \in C^1[c_i, d_i], \quad G(t) \geq 0, G(t) \neq 0, G(c_i) = G(d_i) = 0,$$
$$G'(t) = 2g(t)\sqrt{G(t)}, \quad g(t) \in C[c_i, d_i], i = 1,2\}.$$

Lemma 2.1. *If the impulsive differential inequality*

$$\left.\begin{aligned} &[r(t)g(v')]' + \varepsilon q(t)v(t-\rho) + \sum_{j=1}^{m} \varepsilon_j q_j(t)v(t-\rho) \leq F(t), \quad t \neq t_k \\ &\alpha_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq \alpha_k \ ; \quad \beta_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq \beta_k, \quad k = 1,2,\cdots, \end{aligned}\right\} \quad (2.1)$$

has no eventually positive solution, then every solution of the boundary value problem defined by (1.1)-(1.2) is oscillatory in G.

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x,t)$ of the boundary value problem $(1.1) - (1.2)$. Without loss of generality, we may assume that $u(x,t) > 0$ in $\Omega \times [t_0, +\infty)$ for some $t_0 > 0$, $u(x, t-\rho) > 0$ and $u(x, t - \tau_s) > 0$, $s = 1,2,\cdots,l$.

For $t \neq t_k, t \geq t_0, k = 1,2,\cdots$, we multiplying both sides of equation (1.1) by $K_\Phi\Phi(x) > 0$ and integrating with respect to x over the domain Ω, we obtain

$$\left.\begin{aligned} &\frac{d}{dt}\left[r(t)g\left[\frac{d}{dt}\int_\Omega u(x,t)K_\Phi\Phi(x)dx\right]\right] + K_\Phi \int_\Omega q(x,t)f\big(u(x,t-\rho)\big)\Phi(x)dx \\ &+K_\Phi \int_\Omega q_j(x,t)f_j\big(u(x,t-\rho)\big)\Phi(x)dx = a(t)K_\Phi \int_\Omega \Delta u(x,t)\Phi(x)dx \\ &+\sum_{s=1}^{l} a_s(t)K_\Phi \int_\Omega \Delta u(x,t-\tau_s)\Phi(x)dx + K_\Phi \int_\Omega F(x,t)\Phi(x)dx. \end{aligned}\right\} \quad (2.2)$$

From Green's formula and the boundary condition (1.2), we see that

$$K_\Phi \int_\Omega \Delta u(x,t)\Phi(x)dx = K_\Phi \int_{\partial\Omega}\left[\Phi(x)\frac{\partial u}{\partial\gamma} - u\frac{\partial\Phi(x)}{\partial\gamma}\right]dS + K_\Phi \int_\Omega u(x,t)\Delta\Phi(x)dx$$
$$= 0 - \lambda_0 v(t) \leq 0, \quad (2.3)$$

and for $s = 1,2,\cdots,l$, we have

$$K_\Phi \int_\Omega \Delta u(x,t-\tau_s)\Phi(x)dx = K_\Phi \int_{\partial\Omega}\left[\Phi(x)\frac{\partial u(x,t-\tau_s)}{\partial\gamma} - u(x,t-\rho_s)\frac{\partial\Phi(x)}{\partial\gamma}\right]dS$$
$$+K_\Phi \int_\Omega u(x,t-\tau_s)\Delta\Phi(x)dx$$
$$= 0 - \lambda_0 v(t-\tau_s) \le 0, \tag{2.4}$$

where dS is surface component on $\partial\Omega$. Furthermore applying Jensen's inequality for convex functions and using the assumptions on (H_1), we get that

$$K_\Phi \int_\Omega q(x,t)f(u(x,t-\rho))\Phi(x)dx \ge q(t)K_\Phi \int_\Omega f(u(x,t-\rho))\Phi(x)dx$$
$$\ge \varepsilon q(t)v(t-\rho), \tag{2.5}$$

and for $j = 1,2,\cdots,m$

$$K_\Phi \int_\Omega q_j(x,t)f_j\big(u(x,t-\rho)\big)\Phi(x)dx \ge q_j(t)K_\Phi \int_\Omega f_j\big(u(x,t-\rho)\big)\Phi(x)dx$$
$$\ge \varepsilon_j q_j(t)v(t-\rho). \tag{2.6}$$

Take

$$F(t) = K_\Phi \int_\Omega F(x,t)\Phi(x)dx. \tag{2.7}$$

Combining (2.2)-(2.7), we get that

$$\left[r(t)g(v'(t))\right]' + \varepsilon q(t)v(t-\rho) + \sum_{j=1}^{m} \varepsilon_j q_j(t)v(t-\rho) \le F(t).$$

For $t = t_k,\ k = 1,2,\cdots$, multiplying both sides of the equation (1.1) by $K_\Phi\Phi(x) > 0$, integrating with respect to x over the domain Ω, and from (H_5), we obtain

$$\alpha_k^* \le \frac{u(x,t_k^+)}{u(x,t_k)} \le \alpha_k,\quad \beta_k^* \le \frac{\frac{\partial u(x,t_k^+)}{\partial t}}{\frac{\partial u(x,t_k)}{\partial t}} \le \beta_k.$$

Since $v(t) = K_\Phi \int_\Omega u(x,t)\Phi(x)dx$, we have

$$\alpha_k^* \le \frac{v(t_k^+)}{v(t_k)} \le \alpha_k,\quad \beta_k^* \le \frac{v'(t_k^+)}{v'(t_k)} \le \beta_k.$$

Therefore $v(t)$ is an eventually positive solution of (2.1), which contradicts the hypothesis and completes the proof. ■

Theorem 2.1 *Assume that conditions $(H_1) - (H_5)$ hold, furthermore for any $T \ge 0$ there exist c_i, d_i satisfying (H_6) with $T \le c_1 < d_1$, $T \le c_2 < d_2$ and $p(t) \in J_p(c_1,d_1)$ such that*

$$\int_{c_i}^{t_{I(c_i)+1}} Q(t)p^2(t)M_{I(c_i)}^i(t)dt + \sum_{k=I(c_i)+1}^{I(d_i)-1}\int_{t_k}^{t_{k+1}} Q(t)p^2(t)M_k^i(t)dt$$
$$+\int_{t_{I(d_i)}}^{d_i} Q(t)p^2(t)M_{I(d_i)}^i(t)dt - \int_{c_i}^{d_i}\eta r(t)(p'(t))^2dt \ge \Lambda(p,c_i,d_i) \tag{2.8}$$

where $\Lambda(p,c_i,d_i) = 0$ for $I(c_i) = I(d_i)$ and

$$\Lambda(p,c_i,d_i) = r_i\left\{p^2(t_{I(c_i)+1})\frac{\beta_{I(c_i)+1} - \alpha_{I(c_i)+1}^*}{\alpha_{I(c_i)+1}^*(t_{I(c_i)+1} - c_i)} + \sum_{k=I(c_i)+2}^{I(d_i)} p^2(t_k)\frac{\beta_k - \alpha_k^*}{\alpha_k^*(t_k - t_{k-1})}\right\}$$

for $I(c_i) < I(d_i)$, $i = 1,2$

$$M_k^i(t) = \begin{cases} \frac{1}{\eta}\frac{t-t_k}{\alpha_k\rho+\beta_k(t-t_k)}, & t\in(t_k,t_k+\rho) \\ \frac{1}{\eta}\frac{t-\rho-t_k}{t-t_k}, & t\in[t_k+\rho,t_{k+1}), \end{cases}$$

then every solution of the boundary value problem $(1.1)-(1.2)$ is oscillatory in G.

Proof. Assume to the contrary that $v(t)$ is a non-oscillatory solution of (2.1). Without loss of generality we may assume that $v(t)$ is an eventually positive solution of (2.1). Then there exists $t_1 \geq t_0$ such that $v(t) > 0$ for $t \geq t_1$. Therefore it follows from (2.1) that

$$\left[r(t)g(v^{'}(t))\right]^{'} \leq F(t) - \varepsilon q(t)v(t-\rho) - \sum_{j=1}^{m} \varepsilon_j q_j(t)v(t-\rho) \leq 0 \quad \text{for} t\in[t_1,+\infty). \quad (2.9)$$

Thus $v^{'}(t) \geq 0$ or $v^{'}(t) < 0$, $t \geq t_1$ for some $t_1 \geq t_0$. We now claim that

$$v^{'}(t) \geq 0 \quad \text{for} t \geq t_1. \quad (2.10)$$

Suppose not, then $v^{'}(t) < 0$ and there exists $t_2 \in [t_1,+\infty)$ such that $v^{'}(t_2) < 0$. Since $r(t)g(v^{'}(t))$ is strictly decreasing on $[t_1,+\infty)$. It is clear that

$$r(t)g(v^{'}(t)) < r(t_2)g(v^{'}(t_2)) := -\mu$$

where $\mu > 0$ is a constant for $t \in [t_2,+\infty)$, we have

$$r(t)g(v^{'}(t)) < -\mu$$

$$v^{'}(t) < g^{-1}\left(\frac{-\mu}{r(t)}\right)$$

$$v^{'}(t) \leq -\zeta_1 g^{-1}\left(\frac{1}{r(t)}\right), \quad \text{where} \zeta_1 = \zeta g^{-1}(\mu) \quad \text{for} t\in[t_2,+\infty).$$

Integrating the above inequality from t_2 to t, we have

$$v(t) \leq v(t_2) - \zeta_1 \int_{t_2}^{t} g^{-1}\left(\frac{1}{r(s)}\right) ds.$$

Letting $t \to +\infty$, we get

$$\lim_{t\to+\infty} v(t) = -\infty$$

which contradiction proves that (2.10) holds. Define the Riccati Transformation

$$w(t) := -\frac{r(t)g\left(v^{'}(t)\right)}{v(t)}. \quad (2.11)$$

It follows from (2.1) that $w(t)$ satisfies

$$w^{'}(t) \geq -\frac{F(t)}{v(t)} + \left[\varepsilon q(t) + \sum_{j=1}^{m} \varepsilon_j q_j(t)\right]\frac{v(t-\rho)}{v(t)} + \frac{w^2(t)}{\eta r(t)}.$$

By the assumption, we can choose $c_1, d_1 \geq t_0$ such that $r(t) \geq 0$, $q(t) \geq 0$ and $q_j(t) \geq 0$ for $t \in [c_1-\rho, d_1]$, $j = 1,2,\cdots,m$ and $F(t) \leq 0$ for $t \in [c_1-\rho, d_1]$ from (2.1) we can easily to see that

$$w^{'}(t) \geq \frac{w^2(t)}{\eta r(t)} + Q(t)\frac{v(t-\rho)}{v(t)}. \quad (2.12)$$

For $t = t_k,\ k = 1,2,\cdots$, one has

$$w(t_k^+) = -\frac{r(t_k^+)g\left(v^{'}(t_k^+)\right)}{v(t_k^+)} \geq \frac{\beta_k}{\alpha_k^*} w(t_k). \quad (2.13)$$

At first, we consider the case in which $I(c_1) < I(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{I(c_1)+1},\ t_{I(c_1)+2},\cdots,\ t_{I(d_1)}$. Choose an $p(t) \in J_p(c_1, d_1)$ and multiplying by $p^2(t)$ on both sides on (2.12), integrating it from c_1 to d_1, we obtain

$$\int_{c_1}^{t_{I(c_1)+1}} p^2(t)w^{'}(t)dt + \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} p^2(t)w^{'}(t)dt + \cdots + \int_{t_{I(d_1)}}^{d_1} p^2(t)w^{'}(t)dt$$

$$\geq \int_{c_1}^{t_{I(c_1)+1}} p^2(t)\frac{w^2(t)}{\eta r(t)}dt + \int_{t_{I(c_1)+1}}^{t_{I(c_1)+2}} p^2(t)\frac{w^2(t)}{\eta r(t)}dt + \cdots + \int_{t_{I(d_1)}}^{d_1} p^2(t)\frac{w^2(t)}{\eta r(t)}dt$$
$$+\int_{c_1}^{t_{I(c_1)+1}} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt + \int_{t_{I(c_1)+1}}^{t_{I(c_1)+1}+\rho} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt$$
$$+\int_{t_{I(c_1)+1}+\rho}^{t_{I(c_1)+2}} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt$$
$$+\cdots+\int_{t_{I(c_1)-1}+\rho}^{t_{I(d_1)}} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt + \int_{t_{I(d_1)}}^{d_1} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt.$$

Using the integration by parts on the left-hand side, and noting that the condition$p(c_1) = p(d_1) = 0$, we get

$$\sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)[w(t_k) - w(t_k^+)] \geq \int_{c_1}^{t_{I(c_1)+1}} \frac{\eta}{r(t)}\left[r(t)p'(t) + \frac{p(t)w(t)}{\eta}\right]^2 dt$$
$$+\sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \frac{\eta}{r(t)}\left[r(t)p'(t) + \frac{p(t)w(t)}{\eta}\right]^2 dt + \int_{t_{I(d_1)}}^{d_1} \frac{\eta}{r(t)}\left[r(t)p'(t) + \frac{p(t)w(t)}{\eta}\right]^2 dt$$
$$+\int_{c_1}^{t_{I(c_1)+1}} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt$$
$$+\sum_{k=I(c_1)+1}^{I(d_1)-1}\left[\int_{t_k}^{t_k+\rho} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt + \int_{t_k+\rho}^{t_{k+1}} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt\right]$$
$$+\int_{t_{I(d_1)}}^{d_1} p^2(t)Q(t)\frac{v(t-\rho)}{v(t)}dt - \int_{c_1}^{d_1} \eta r(t)(p'(t))^2 dt. \qquad (2.14)$$

There are several cases to consider to estimate $\frac{v(t-\rho)}{v(t)}$.

Case 1: For $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$. If $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$, since $t_{k+1} - t_k > \rho$, we consider two sub cases:

Case 1.1: If $t \in [t_k + \rho, t_{k+1}]$, then $t - \rho \in [t_k, t_{k+1} - \rho]$ and there are no impulsive moments in $(t-\rho, t)$, then for any $t \in [t_k + \rho, t_{k+1}]$ one has

$$v(t) - v(t_k^+) = v'(\xi_1)(t - t_k), \quad \xi_1 \in (t_k, t).$$

Since $r(t)g(v'(t))$ is non-increasing

$$v(t) \geq v'(\xi_1)(t - t_k) > \frac{r(t)g'(t)}{r(\xi_1)}(t - t_k).$$

From the fact that $r(t)$ is nondecreasing, we get

$$\frac{r(t)g(v'(t))}{v(t)} < \frac{r(\xi_1)}{t - t_k} < \frac{r(t)}{t - t_k}.$$

We obtain

$$\frac{v'(t)}{v(t)} < \frac{1}{\eta}\frac{1}{t - t_k}.$$

Integrating it from $t - \rho$ to t, we have

$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t - \rho - t_k}{t - t_k}.$$

Case 1.2: If $t \in (t_k, t_k + \rho)$ then $t - \rho \in (t_k - \rho, t_k)$ and there is an impulsive moment t_k in $(t-\rho, t)$. Similar to Case 1.1, we obtain

$$v(t) - v(t_k - \rho) = v'(\xi_2)(t - t_k + \rho), \quad \xi_2 \in (t_k - \rho, t_k]$$

or

$$\frac{v'(t)}{v(t)} < \frac{1}{\eta}\frac{1}{t - t_k + \rho}.$$

Integrating it from $t-\rho$ to t, we get
$$\frac{v(t-\rho)}{v(t_k)} > \frac{1}{\eta}\frac{t-t_k}{\rho} \geq 0, \quad t \in (t_k, t_k+\rho). \; (2.15)$$
For any $t \in (t_k, t_k+\rho)$, we have
$$v(t) - v(t_k^+) < v'(t_k^+)(t-t_k).$$
Using the impulsive conditions in equation (2.1), we get
$$v(t) - \alpha_k v(t_k) < \beta_k v'(t_k)(t-t_k)$$
$$\frac{v(t)}{v(t_k)} < \frac{v'(t_k)}{v(t_k)}\beta_k(t-t_k) + \alpha_k.$$
Using $\frac{v'(t_k)}{v(t_k)} < \frac{1}{\rho}$, we obtain
$$\frac{v(t)}{v(t_k)} < \alpha_k + \frac{1}{\rho}\beta_k(t-t_k).$$
That is,
$$\frac{v(t_k)}{v(t)} > \frac{\rho}{\alpha_k\rho + \beta_k(t-t_k)}. \qquad (2.16)$$
From (2.15) and (2.16), we get
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-t_k}{\alpha_k\rho + \beta_k(t-t_k)} \geq 0.$$
Case 2: If $t \in [c_1, t_{I(c_1)+1}]$, we consider three sub cases:

Case 2.1: If $t_{I(c_1)} > c_1 - \rho$ and $t \in [t_{I(c_1)}+\rho, t_{I(c_1)+1}]$ then $t-\rho \in [t_{I(c_1)}, t_{I(c_1)+1}-\rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 1.1 and using Mean-value Theorem on $(t_{I(c_1)}, t_{I(c_1)+1}]$, we get
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-\rho-t_{I(c_1)}}{t-t_{I(c_1)}} \geq 0.$$
Case 2.2: If $t_{I(c_1)} > c_1 - \rho$ and $t \in [c_1, t_{I(c_1)}+\rho)$, then $t-\rho \in [c_1-\rho, t_{I(c_1)})$ and there is an impulsive moments $t_{I(c_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 1.2, we have
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-t_{I(c_1)}}{\alpha_{I(c_1)}\rho + \beta_{I(c_1)}(t-t_{I(c_1)})} \geq 0.$$
Case 2.3: If $t_{I(c_1)} < c_1 - \rho$, then for any $t \in [c_1, t_{I(c_1)+1}]$, $t-\rho \in [c_1-\rho, t_{I(c_1)+1}-\rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 1.1, we obtain
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-\rho-t_{I(c_1)}}{t-t_{I(c_1)}} \geq 0.$$
Case 3: For $t \in (t_{I(d_1)}, d_1]$, there are three sub cases:

Case 3.1: If $t_{I(d_1)}+\rho < d_1$ and $t \in [t_{I(d_1)}+\rho, d_1]$ then $t-\rho \in [t_{I(d_1)}, d_1-\rho]$ and there are no impulsive moments in $(t-\rho, t)$. Making a similar analysis of the Case 2.1, we have
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-\rho-t_{I(d_1)}}{t-t_{I(d_1)}} \geq 0.$$
Case 3.2: If $t_{I(d_1)}+\rho < d_1$ and $t \in [t_{I(d_1)}, t_{I(d_1)}+\rho)$, then $t-\rho \in [t_{I(d_1)}-\rho, t_{I(d_1)})$ and there is an impulsive moments $t_{I(d_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 2.2, we obtain
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-t_{I(d_1)}}{\alpha_{I(d_1)}\rho + \beta_{I(d_1)}(t-t_{I(d_1)})} \geq 0.$$
Case 3.3: If $t_{I(d_1)}+\rho \geq d_1$, then for any $t \in (t_{I(d_1)}, d_1]$, we get $t-\rho \in (t_{I(d_1)}-\rho, d_1-\rho]$ and there is an impulsive moments $t_{I(d_1)}$ in $(t-\rho, t)$. Making a similar analysis of the Case 3.2, we get
$$\frac{v(t-\rho)}{v(t)} > \frac{1}{\eta}\frac{t-t_{I(d_1)}}{\alpha_{I(d_1)}\rho + \beta_{I(d_1)}(t-t_{I(d_1)})} \geq 0.$$
Combining all these cases, we have

$$\frac{v(t-\rho)}{v(t)} > \begin{cases} M^1_{I(c_1)}(t) & for t \in [c_1, t_{I(c_1)+1}] \\ M^1_k(t) & for t \in (t_k, t_{k+1}], \quad k = I(c_1)+1, \cdots, I(d_1)-1 \\ M^1_{I(d_1)}(t) & for t \in (t_{I(d_1)-1}, d_1]. \end{cases}$$

Hence by (2.14), we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)[w(t_k) - w(t_k^+)]$$

$$\geq \int_{c_1}^{t_{I(c_1)+1}} p^2(t)Q(t)M^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} p^2(t)Q(t)M^1_k(t)dt$$

$$+ \int_{t_{I(d_1)}}^{d_1} p^2(t)Q(t)M^1_{I(d_1)}(t)dt - \int_{c_1}^{d_1} \eta r(t)(p'(t))^2 dt. \qquad (2.17)$$

Since $[r(t)g(v')]' < 0$ for all $t \in (c_1, t_{I(c_1)+1}]$, $r(t)g(v'(t))$ is non-increasing in $(c_1, t_{I(c_1)+1}]$. Thus

$$v(t) > v(t) - v(c_1) = v'(\xi_4)(t - c_1) \geq \frac{r(t)g(v'(t))}{r(\xi_4)}(t - c_1), \quad \xi_4 \in (c_1, t)$$

and hence $\frac{r(t)g(v'(t))}{v(t)} < \frac{r(\xi_4)}{t-c_1}$. Letting $t \to t^-_{I(c_1)+1}$, it follows that

$$w(t_{I(c_1)+1}) \geq -\frac{r_1}{t_{I(c_1)+1} - c_1}. \qquad (2.18)$$

Similarly we can prove that on $(t_{k-1}, t_k]$, $k = I(c_1)+2, \cdots, I(d_1)$,

$$w(t_k) \geq -\frac{r_1}{t_k - t_{k-1}}. \qquad (2.19)$$

Hence (2.18) and (2.19), we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)w(t_k)\left[\frac{\beta_k - \alpha_k^*}{\alpha_k^*}\right] \geq -r_1\left[p^2(t_{I(c_1)+1})\frac{\beta_{I(c_1)+1} - \alpha^*_{I(c_1)+1}}{\alpha^*_{I(c_1)+1}}\frac{1}{t_{I(c_1)+1} - c_1}\right.$$

$$\left.+ \sum_{k=I(c_1)+2}^{I(d_1)} p^2(t_k)\frac{\beta_k - \alpha_k^*}{\alpha_k^*}\frac{1}{t_k - t_{k-1}}\right]$$

$\geq -\Lambda(p, c_1, d_1)$.

Thus we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} p^2(t_k)w(t_k)\left[\frac{\alpha_k^* - \beta_k}{\alpha_k^*}\right] \leq \Lambda(p, c_1, d_1).$$

Therefore (2.17), we get

$$\int_{c_1}^{t_{I(c_1)+1}} p^2(t)Q(t)M^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} p^2(t)Q(t)M^1_k(t)dt$$

$$+ \int_{t_{I(d_1)}}^{d_1} p^2(t)Q(t)M^1_{I(d_1)}(t)dt - \int_{c_1}^{d_1} \eta r(t)(p'(t))^2 dt < \Lambda(p, c_1, d_1)$$

which contradicts (2.8).

If $I(c_1) = I(d_1)$ then $\Lambda(p, c_1, d_1) = 0$ and there are no impulsive moments in $[c_1, d_1]$. Similar to the proof of (2.17), we obtain

$$\int_{c_1}^{d_1} [p^2(t)Q(t)M^1_{I(c_1)}(t) - \eta r(t)(p'(t))^2]dt < 0.$$

This again contradicts our assumption. Finally if $v(t)$ is eventually negative, we can consider $[c_2, d_2]$ and reach similar contradiction. The proof of theorem is complete. ■

Theorem 2.2. *Suppose that* $(H_1)-(H_5)$ *hold, furthermore for any* T_0 *there exist* c_i, d_i *satisfying* (H_6) *with* $T \le c_1 < d_1$, $T \le c_2 < d_2$ *and* $G(t) \in J_G(c_i, d_i)$ *such that*

$$\int_{c_i}^{t_{I(c_i)+1}} G(t)Q(t)M^i_{I(c_i)}(t)dt + \sum_{k=I(c_i)+1}^{I(d_i)-1} \int_{t_k}^{t_{k+1}} G(t)Q(t)M^i_k(t)dt + \int_{t_{I(d_i)}}^{d_i} G(t)Q(t)M^i_{I(d_i)}(t)dt - \int_{c_i}^{d_i} \eta r(t)g^2(t)dt \ge \Theta(G, c_i, d_i) \quad (2.20)$$

where $\Theta(G, c_i, d_i) = 0$ for $I(c_i) = I(d_i)$ and

$$\Theta(G, c_i, d_i) = r_i \left\{ G(t_{I(c_i)+1}) \frac{\beta_{I(c_i)+1} - \alpha^*_{I(c_i)+1}}{\alpha^*_{I(c_i)+1}(t_{I(c_i)+1} - c_i)} + \sum_{k=I(c_i)+2}^{I(d_i)} G(t_k) \frac{\beta_k - \alpha^*_k}{\alpha^*_k (t_k - t_{k-1})} \right\}$$

for $I(c_i) < I(d_i)$, $i = 1,2$, then every solution of the boundary value problem $(1.1)-(1.2)$ is oscillatory in G.

Proof. Similar to the proof of Theorem 2.1, suppose $v(t-\rho)$ for $t \ge t_0$. If $I(c_i) < I(d_i)$, multiplying $G(t)$ throughout (2.12) and integrating over $[c_i, d_i]$, we get

$$\sum_{k=I(c_1)+1}^{I(d_1)} G(t_k)[w(t_k) - w(t_k^+)] \ge \int_{c_1}^{t_{I(c_1)+1}} \frac{1}{\eta}\left[\sqrt{\frac{G(t)}{r(t)}} w(t) + \eta g(t)\sqrt{r(t)}\right]^2 dt$$

$$+ \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} \frac{1}{\eta}\left[\sqrt{\frac{G(t)}{r(t)}} w(t) + \eta g(t)\sqrt{r(t)}\right]^2 dt + \int_{t_{I(d_1)}}^{d_1} \frac{1}{\eta}\left[\sqrt{\frac{G(t)}{r(t)}} w(t) + \eta g(t)\sqrt{r(t)}\right]^2 dt$$

$$+ \int_{c_1}^{t_{I(c_1)+1}} G(t)Q(t)\frac{v(t-\rho)}{v(t)} dt$$

$$+ \sum_{k=I(c_1)+1}^{I(d_1)-1} \left[\int_{t_k}^{t_k+\rho} G(t)Q(t)\frac{v(t-\rho)}{v(t)} dt + \int_{t_k+\rho}^{t_{k+1}} G(t)Q(t)\frac{v(t-\rho)}{v(t)} dt\right]$$

$$+ \int_{t_{I(d_1)}}^{d_1} G(t)Q(t)\frac{v(t-\rho)}{v(t)} dt - \int_{c_1}^{d_1} \eta r(t)g^2(t)dt.$$

$$\ge \int_{c_1}^{t_{I(c_1)+1}} G(t)Q(t)M^1_{I(c_1)}dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} G(t)Q(t)M^1_k dt + \int_{t_{I(d_1)}}^{d_1} G(t)Q(t)M^1_{I(d_1)}dt - \int_{c_1}^{d_1} \eta r(t)g^2(t)dt. \quad (2.21)$$

On the other hand, from the proof of Theorem 2.1, we have

$$w(t_{I(c_1)+1}) \ge -\frac{r_1}{t_{I(c_1)+1} - c_1}, \quad w(t_k) \ge -\frac{r_1}{t_k - t_{k-1}}.$$

for $k = I(c_1)+2, \cdots, I(d_1)$. We get

$$\sum_{k=I(c_1)+1}^{I(d_1)} G(t_k)w(t_k)\left[\frac{\beta_k - \alpha^*_k}{\alpha^*_k}\right] \ge -r_1 \left[G(t_{I(c_1)+1}) \frac{\beta_{I(c_1)+1} - \alpha^*_{I(c_1)+1}}{\alpha^*_{I(c_1)+1}} \frac{1}{t_{I(c_1)+1} - c_1} + \sum_{k=I(c_1)+2}^{I(d_1)} G(t_k)\frac{\beta_k - \alpha^*_k}{\alpha^*_k}\frac{1}{t_k - t_{k-1}}\right]$$

$$\ge -\Theta(G, c_1, d_1).$$

Thus we have

$$\sum_{k=I(c_1)+1}^{I(d_1)} G(t_k)w(t_k)\left[\frac{\alpha^*_k - \beta_k}{\alpha^*_k}\right] \le \Theta(G, c_1, d_1).$$

Therefore (2.21), we get

$$\int_{c_1}^{t_{I(c_1)+1}} G(t)Q(t)M^1_{I(c_1)}(t)dt + \sum_{k=I(c_1)+1}^{I(d_1)-1} \int_{t_k}^{t_{k+1}} G(t)Q(t)M^1_k(t)dt + \int_{t_{I(d_1)}}^{d_1} G(t)Q(t)M^1_{I(d_1)}(t)dt - \int_{c_1}^{d_1} \eta r(t)g^2(t)dt < \Theta(G, c_1, d_1)$$

which contradicts (2.20). If $I(c_1) = I(d_1)$, the proof is similar to that the Theorem 2.1, and so it is omitted here. The proof of theorem is complete. ■

Next, we will establish Kemenev type oscillation criteria for (1.1) following the ideas of [6] and [13]. Let $D = \{(t,s): t_0 \le s \le t\}$, then a function $H \in C(D, \mathbb{R})$ is said to belong to the class $\mathcal{H}$ if

$(H_7) H(t,t) = 0$, $H(t,s) > 0$ for $t > s$ and

$(H_8) H$ has partial derivative $\frac{\partial H}{\partial t}$ and $\frac{\partial H}{\partial s}$ on D such that

$$\frac{\partial H}{\partial t} = 2h_1(t,s)\sqrt{H(t,s)}, \quad \frac{\partial H}{\partial s} = -2h_2(t,s)\sqrt{H(t,s)}$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

The following two lemmas are needed to prove our theorem.

Lemma 2.2. *Suppose that $(H_1) - (H_5)$ hold and $v(t)$ is a solution of (1.1)-(1.2). If there exist $\theta_i \in (c_i, d_i)$, $\theta_i \notin \{t_k\}$, $i = 1,2$ such that $v(t) > 0$ on $[\theta_1, d_1]$ and $v(t) < 0$ on $[\theta_2, d_2]$ then for any $H \in \mathcal{H}$*

$$\int_{\theta_i}^{t_{I(\theta_i)+1}} H(d_i,s)Q(s)M^i_{I(\theta_i)}(t)dt + \sum_{k=I(\theta_i)+1}^{I(d_i)-1} \int_{t_k}^{t_{k+1}} H(d_i,s)Q(s)M^i_k(t)dt + \int_{t_{I(d_i)}}^{d_i} H(d_i,s)Q(s)M^i_{I(d_i)}(t)dt \le \sum_{k=I(\theta_i)}^{I(d_i)} H(d_i,t_k)\frac{\alpha^*_k - \beta_k}{\alpha^*_k}w(t_k) - H(d_i,\theta_i)w(\theta_i) + \int_{\theta_i}^{d_i} \eta r(s)h_2^2(d_i,s)ds. \quad (2.22)$$

Proof. The proof is similar to [5]. So we omit it. ■

Lemma 2.3. *Suppose that $(H_1) - (H_5)$ hold and $v(t)$ is a solution of (1.1)-(1.2). If there exist $\theta_i \in (c_i, d_i)$, $\theta_i \notin \{t_k\}$, $i = 1,2$ such that $v(t) > 0$ on $[c_1, \theta_1)$ and $v(t) < 0$ on $[c_2, \theta_2)$ then for any $H \in \mathcal{H}$*

$$\int_{c_i}^{t_{I(c_i)+1}} H(s,c_i)Q(s)M^i_{I(c_i)}(t)dt + \sum_{k=I(c_i)+1}^{I(\theta_i)-1} \int_{t_k}^{t_{k+1}} H(s,c_i)Q(s)M^i_k(t)dt + \int_{t_{I(\theta_i)}}^{\theta_i} H(s,c_i)Q(s)M^i_{I(\theta_i)}(t)dt \le \sum_{k=I(c_i)}^{I(\theta_i)} H(t_k,c_i)\frac{\alpha^*_k - \beta_k}{\alpha^*_k}w(t_k) - H(\theta_i,c_i)w(\theta_i) + \int_{c_i}^{\theta_i} \eta r(s)h_1^2(s,c_i)ds. \quad (2.23)$$

Proof. The proof is similar to [5]. So we omit it. ■

Similar to [[9], Theorem 2.3], we have the following theorem

Theorem 2.3. *Suppose that $(H_1) - (H_5)$ hold. Assume that there are $\theta_i \in (c_i, d_i)$, $i = 1,2$, and $H \in \mathcal{H}$ such that*

$$\frac{1}{H(d_i,\theta_i)}\left[\int_{\theta_i}^{t_{I(\theta_i)+1}} H(d_i,s)Q(s)M^i_{I(\theta_i)}(t)dt + \sum_{k=I(\theta_i)+1}^{I(d_i)-1} \int_{t_k}^{t_{k+1}} H(d_i,s)Q(s)M^i_k(t)dt\right.$$

$$+\int_{t_{I(d_i)}}^{d_i} H(d_i,s)Q(s)M^i_{I(d_i)}(t)dt-\int_{\theta_i}^{d_i}\eta r(s)h_2^2(d_i,s)ds\Bigg]$$

$$+\frac{1}{H(\theta_i,c_i)}\Bigg[\int_{c_i}^{t_{I(c_i)+1}} H(s,c_i)Q(s)M^i_{I(c_i)}(t)dt+\sum_{k=I(c_i)+1}^{I(\theta_i)-1}\int_{t_k}^{t_{k+1}} H(s,c_i)Q(s)M^i_k(t)dt$$

$$+\int_{t_{I(\theta_i)}}^{\theta_i} H(s,c_i)Q(s)M^i_{I(\theta_i)}(t)dt-\int_{c_i}^{\theta_i}\eta r(s)h_1^2(s,c_i)ds\Bigg]$$

$$\geq\ \Xi(H,c_i,d_i), \tag{2.24}$$

where $\Xi(H,c_i,d_i)=0$ for $I(c_i)=I(d_i)$ and

$$\Xi(H,c_i,d_i)=$$

$$\frac{r_i}{H(d_i,\theta_i)}\Bigg[H\big(d_i,t_{I(\theta_i)+1}\big)\frac{\beta_{I(\theta_i)+1}-\alpha^*_{I(\theta_i)+1}}{\alpha^*_{I(\theta_i)+1}\big(t_{I(\theta_i)+1}-\theta_i\big)}+\sum_{k=I(\theta_i)+2}^{I(d_i)} H(d_i,t_k)\frac{\beta_k-\alpha_k^*}{\alpha_k^*(t_k-t_{k-1})}\Bigg]$$

$$+\frac{r_i}{H(\theta_i,c_i)}\Bigg[H(t_{I(c_i)+1},c_i)\frac{\beta_{I(c_i)+1}-\alpha^*_{I(c_i)+1}}{\alpha^*_{I(c_i)+1}(t_{I(c_i)+1}-c_i)}+\sum_{k=I(c_i)+2}^{I(\theta_i)} H(t_k,c_i)\frac{\beta_k-\alpha_k^*}{\alpha_k^*(t_k-t_{k-1})}\Bigg]$$

for $I(c_i)<I(d_i)$, $i=1,2$ then every solution of (1.1)-(1.2) is oscillatory in G.

3. Example

In this section, we present an example to illustrate our results established in Section 2.

Example 1 *Consider the following impulsive partial differential equation*

$$\left.\begin{aligned}&\frac{\partial}{\partial t}\left[3g\left(\frac{\partial}{\partial t}u(x,t)\right)\right]+mu\left(x,t-\frac{\pi}{8}\right)+4u\left(x,t-\frac{\pi}{8}\right)\\&=6\Delta u(x,t)+5\Delta u\left(x,t-\frac{\pi}{8}\right)+F(x,t),\quad t\neq 2k\pi\pm\frac{\pi}{4},\\&u(x,t_k^+)=\frac{1}{3}u(x,t_k),\quad u_t(x,t_k^+)=\frac{2}{3}u_t(x,t_k),\qquad k=1,2,\cdots,\end{aligned}\right\} \tag{3.1}$$

for $(x,t)\in(0,\pi)\times\mathbb{R}_+$,with the boundary condition

$$u(0,t)=u(\pi,t)=0,\quad t\neq 2k\pi\pm\frac{\pi}{4},\quad k=1,2,\cdots. \tag{3.2}$$

Here $\Omega=(0,\pi)$, $N=1$, $\alpha_k=\alpha_k^*=\frac{1}{3}$, $\beta_k=\beta_k^*=\frac{2}{3}$, $r(t)=3$, $q(t)=m$, $q_1(t)=4$, $g(u)=2u$, $f(u)=f_1(u)=u$, $a(t)=6$, $a_1(t)=5$, $\tau_1=\frac{\pi}{8}$, $\eta=2$, $F(x,t)=9\sin x\cos\left(t-\frac{\pi}{8}\right)+m\sin x\cos\left(t-\frac{\pi}{8}\right)$ and m is a positive constant. Also $\rho=\frac{\pi}{8}$, $t_{k+1}-t_k=\pi/2>\pi/8$. For any $T>0$, we choose k large enough such that $T<c_1=4k\pi-\frac{\pi}{2}<d_1=4k\pi$ and $c_2=4k\pi+\frac{\pi}{8}<d_2=4k\pi+\frac{\pi}{2}$, $k=1,2,\cdots$. Then there is an impulsive movement $t_k=4k\pi-\frac{\pi}{4}$ in $[c_1,d_1]$ and an impulsive moment $t_{k+1}=4k\pi+\frac{\pi}{4}$ in $[c_2,d_2]$. For $\varepsilon=\varepsilon_1=1$, we have $Q(t)=m+4$, and we take $p(t)=\sin 16t\in J_p(c_i,d_i)$, $i=1,2$, $t_{I(c_1)}=4k\pi-\frac{7\pi}{4}$, $t_{I(d_1)}=4k\pi-\frac{\pi}{4}$, then by using simple calculation, the left side of Equation (2.8) is the following :

$$\int_{c_1}^{t_{I(c_1)+1}} Q(t)p^2(t)M^1_{I(c_1)}(t)dt+\sum_{k=I(c_1)+1}^{I(d_1)-1}\int_{t_k}^{t_{k+1}} Q(t)p^2(t)M^1_k(t)dt$$

$$+\int_{t_{I(d_1)}}^{d_1} Q(t)p^2(t)M^1_{I(d_1)}(t)dt-\int_{c_1}^{d_1}\eta r(t)(p'(t))^2dt$$

$$\geq(m+4)\left[\int_{4k\pi-\frac{\pi}{2}}^{4k\pi-\frac{\pi}{4}}\sin^2(16t)\left(\frac{t-\frac{\pi}{8}-4k\pi+\frac{7\pi}{4}}{t-4k\pi+\frac{7\pi}{4}}\right)dt\right.$$

$$+\int_{4k\pi-\frac{\pi}{4}}^{4k\pi-\frac{\pi}{8}}\sin^2(16t)\left(\frac{t-4k\pi+\frac{\pi}{4}}{\frac{\pi}{8}\left(\frac{1}{3}\right)+\left(\frac{2}{3}\right)\left(t-4k\pi+\frac{\pi}{4}\right)}\right)dt$$

$$+\int_{4k\pi-\frac{\pi}{8}}^{4k\pi}\sin^2(16t)\left(\frac{t-\frac{\pi}{8}-4k\pi+\frac{\pi}{4}}{t-4k\pi+\frac{\pi}{4}}\right)dt\Bigg]$$

$$-\frac{3}{2}(16)^2\int_{4k\pi-\frac{\pi}{2}}^{4k\pi}(1+\cos 32t)dt$$

$$\simeq (m+4)(0.27685)-603.18578.$$

for m large enough. On the other hand, note that $I(c_1)=k+1, I(d_1)=k, r_1=3$, we have $\Lambda(p,c_i,d_i)=0$. Therefore the condition (2.8) is satisfied in $[c_1,d_1]$. Similarly, we can prove that for $t\in[c_2,d_2]$. Hence by Theorem 2.1, every solution of $(3.1)-(3.2)$ is oscillatory. In fact $u(x,t)=\sin x\cos t$ is one such solution of the problem (3.1)-(3.2).

4. Conclusion

In this paper, we have obtained some sufficient conditions to the impulsive partial differential equations. The improvement factors impulses, delay and forcing term that affect the interval qualitative properties of solution in the sequence of subintervals in $\mathbb{R}_+$, were taken into account together. Our newly obtained results in this paper have improved and extended some of the results already prevailing in the existing literature.

References

[1] D.D. Bainov and P.S. Simenov, Impulsive Differential Equations: Periodic Solutions and Applications, *Longman, Harlow*, 1993.

[2] M.A. El Sayed, An oscillation criteria for forced second order linear differential equations, *Proc. Am Math. Soc.,* **118**(1993), 813-817.

[3] Z. Guo, X. Zhou and W. Wang, Interval oscillation criteria for super-half-linear impulsive differential equations with delay,*J. Appl. Math.,* **(2012)**(2012), 1-22.

[4] M. Huang and W.Z. Feng, Oscillation of second order impulsive delay differential equations withforcing term, *J. Natural Science of Heilongjiang University*, **23**(2006), 452-456 (in Chinese).

[5] M. Huang and W.Z. Feng, Forced oscillations for second order delay differential equations with impulses, *Comput. Math. Appl.,* **59**(2010), 18-30.

[6] Q. Kong, Interval criteria for oscillation of second-order linear differential equation, *J. Math. Anal. Appl.,* **229**(1999), 483-492.

[7] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, Theory of Impulsive Differential Equations, *World Scientific Publishers, Singapore,* 1989.

[8] Q.L. Li and W.S. Cheung, Interval oscillation criteria for second-order forced delay differential equations under impulsive effects, *Electron. J. Qual. Theor. Diff. Eq.,* **2013**(43)(2013), 1-11.

[9] X. Liu and Z. Xu, Oscillation of a forced super-linear second order differential equation with impulses, *Comput. Math. Appl.,* **53**(2007), 1740-1749.

[10] V. Muthulakshmi and E. Thandapani, Interval criteria for oscillation of second-order impulsive differential equation with mixed nonlinearities, *Electron. J. Differ. Eq.*, **2011**(40)(2011), 1-14.

[11] A. Özbekler and A. Zafer, Interval criteria for the forced oscillation of super-half-linear differential equations under impulse effects, *Math. Comput. Model.,* **50**(2009), 59-65.

[12] A. Özbekler and A. Zafer A, Oscillation of solutions of second order mixed nonlinear differential equations under impulsive perturbations, *Comput. Math. Appl.,* **61**(2011), 933-940.

[13] Ch.G. Philos, Oscillation theorems for linear differential equations of second order, *Arch. Math.,* **53**(1989), 482-492.

[14] E. Thandapani, E. Manju and S. Pinelas, Interval oscillation criteria for second order forced impulsive delay differential equations with damping term, *Springer Plus,* **5**(2016), 1-16.

[15] V.S. Vladimirov, Equations of Mathematics Physics, *Nauka, Moscow,* 1981.

[16] J.H. Wu, Theory and Applications of Partial Functional Differential Equations, *Springer, New York,* 1996.

[17] Z. Xiaoliang, G. Zhonghai and W. Wu-Sheng, Interval oscillation criteria for super-half-linear impulsive differential equations with delay, *J. Appl. Math.,* (2012a).
[18] Z. Xiaoliang, G. Zhonghai and W. Wu-Sheng, Interval oscillation criteria of second order mixed nonlinear impulsive differential equations with delay, *Abstr. Appl. Anal.,* (2012a).
[19] N. Yoshida, Oscillation Theory of Partial Differential Equations, *World Scientific, Singapore,* 2008.
[20] C. Zhang, W.Z. Feng, J. Yang and M. Huang, Oscillation of second order impulsive nonlinear FDE with forcing term, *Appl. Math. Comput.,* **198**(2008), 271-279.